

# The invariant joint distribution of a stationary random field and its derivatives: Euler characteristic and critical point counts in 2 and 3D

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The full moments expansion of the joint probability distribution of an isotropic random field, its gradient and invariants of the Hessian is presented in 2 and 3D. It allows for explicit expression for the Euler characteristic in ND and computation of extrema counts as functions of the excursion set threshold and the spectral parameter, as illustrated on model examples.

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Random fields are ubiquitous phenomena in physics appearing in areas from turbulence to the landscape of string theories. In cosmology, the large scale distribution of matter (LSS) and the sky-maps of the polarized Cosmic Microwave Background (CMB) radiation – two focal topics of current research – are described as, respectively, 3D and 2D random fields. Modern view of the Universe, developed primarily through statistical analysis of these fields, points to a Universe that is statistically homogeneous and isotropic with a hierarchy of structures arising from small Gaussian fluctuations of quantum origin.

While the Gaussian limit provides the fundamental starting point in the study of random fields [1–3], non-Gaussian features of the CMB and LSS fields are of great interest. CMB inherits high level of Gaussianity from the initial fluctuations, and small non-Gaussian deviations may provide a unique window into the details of processes in the early Universe. The gravitational instability that nonlinearly maps the initial Gaussian inhomogeneities in matter density into the LSS, on the other hand, induces strong non-Gaussian features culminating in the formation of collapsed, self-gravitating objects such as galaxies and clusters of galaxies. At supercluster scales where non-linearity is mild, the non-Gaussianity of the matter density is also mild, but still essential for quantitative understanding of the filamentary Cosmic Web [4] in-between the galaxy clusters.

The search for the best methods analyzing non-Gaussian random fields both in weak and strong regimes is ongoing. We focus on the statistics of geometrical and topological properties of the field that includes the Euler characteristic of excursion sets [5–8] (and the rest of Minkowski functionals [9]), the density of extremal points and statistics of critical lines, or the skeleton [10, 11]. In this paper we present the formalism for computing such geometrical statistics for the homogeneous and isotropic mildly non-Gaussian fields that can be represented or approximated by a Gram-Charlier expansion around the Gaussian limit, to an arbitrary order of the expansion.

A statistically homogeneous ND random field  $x$  is fully characterized by the joint one-point distribution function (JPDF) of its value and its derivatives  $P(\mathbf{x})$ ,  $\mathbf{x} \equiv (x, x_i, x_{ij}, x_{ijk}, \dots)$ . We consider non-Gaussian fields

$P(\mathbf{x})$  represented by the Gram-Charlier expansion [12]

$$P(\mathbf{x}) = G(\mathbf{x}) \left[ 1 + \sum_{n=3}^{\infty} \frac{1}{n!} \text{Tr} [\langle \mathbf{x}^n \rangle_{\text{GC}} \cdot \mathbf{h}_n(\mathbf{x})] \right] \quad (1)$$

around the Gaussian  $G(\mathbf{x}) \equiv (2\pi)^{-N/2} |\mathbf{C}|^{-1/2} \exp(-\frac{1}{2} \mathbf{x} \cdot \mathbf{C}^{-1} \cdot \mathbf{x})$  that is arranged to match the mean and the covariance  $\mathbf{C} = \langle \mathbf{x} \otimes \mathbf{x} \rangle$  of the  $\mathbf{x}$  variables. We consider all the variables to be defined as having zero mean and unit variance,  $\mathbf{C}_{ii} = 1$ . The correction to the Gaussian approximation is the series in Hermite tensors  $\mathbf{h}_n(\mathbf{x}) = (-1)^n G^{-1}(\mathbf{x}) \partial^n G(\mathbf{x}) / \partial \mathbf{x}^n$  of rank  $n$  with coefficients constructed from the moments,  $\langle \mathbf{x}^n \rangle_{\text{GC}} = \langle \mathbf{h}_n(\mathbf{x}) \rangle$ .

The statistics of Euler characteristic and extremal points [1], and the basic description of the critical lines of the field in the stiff approximation [11], require to know the JPDF only up to the second derivatives. In the Gaussian limit, in this case, the only non-trivial covariance parameter is the cross-correlation between the field and the trace of the Hessian  $\gamma = -\langle x \text{Tr}(x_{ij}) \rangle$  [3, 11].

The JPDF in the form of Eq. (1) is not ideal to study critical sets statistics since the coordinate representation masks the isotropic nature of the statistical descriptors. The pioneering works [6, 7], where the first correction to Euler characteristic was computed, demonstrate the arising complexities. Instead we develop the equivalent of the Gram-Charlier expansion for the JPDF of the field variables that are invariant under coordinate rotation. Such distribution can be computed via explicit integration of the series Eq. (1) over rotations, however we obtain it directly from general principles: *the moment expansion of the non-Gaussian JPDF corresponds to the expansion in the set of polynomials which are orthogonal with respect to the weight provided by the JPDF in the Gaussian limit.* Thus, the problem is reduced to finding such polynomials for a suitable set of invariant variables.

The rotational invariants that are present in the problem are: the field value  $x$  itself, the modulus of its gradient,  $q^2 = \sum_i x_i^2$  and the invariants of the matrix of the second derivatives  $x_{ij}$ . A rank  $N$  symmetric matrix has  $N$  invariants with respect to rotations. The eigenvalues  $\lambda_i$  provide one such representation of invariants, however they are complex algebraic functions of the matrix components. An alternative representation is a set of

invariants that are polynomial in  $x_{ij}$ , with one independent invariant polynomial per order, from one to  $N$ . A familiar example is the set of coefficients,  $I_s$ , of the characteristic equation for the eigenvalues, where the linear invariant is the trace,  $I_1 = \sum_i \lambda_i$ , the quadratic one is  $I_2 = \sum_{i < j} \lambda_i \lambda_j$  and the  $N$ -th order invariant is the determinant of the matrix.  $I_N = \prod_i \lambda_i$ . Aiming at simplifying the JPDF in the Gaussian limit [17]. (e.g., [2, 11], Appendix A) we use their linear combinations  $J_s$

$$J_1 = I_1, \quad J_{s \geq 2} = I_1^s - \sum_{p=2}^s \frac{(-N)^p C_s^p}{(s-1)C_N^p} I_1^{s-p} I_p \quad (2)$$

where  $J_{s \geq 2}$  are (renormalized) coefficients of the charac-

teristic equation of the *traceless part* of the Hessian and are independent in the Gaussian limit on the trace  $J_1$ .

Let us consider the 2D and 3D cases explicitly. Introducing  $\zeta = (x + \gamma J_1)/\sqrt{1 - \gamma^2}$  in place of the field value  $x$  we find that the 2D Gaussian JPDF  $G_{2D}(\zeta, q^2, J_1, J_2)$ , normalized over  $d\zeta dq^2 dJ_1 dJ_2$ , has a fully factorized form

$$G_{2D} = \frac{1}{2\pi} \exp \left[ -\frac{1}{2} \zeta^2 - q^2 - \frac{1}{2} J_1^2 - J_2 \right]. \quad (3)$$

Used as a kernel for the polynomial expansion,  $G_{2D}$  leads to a non-Gaussian rotation invariant JPDF in the form of the direct series in the products of Hermite, for  $\zeta$  and  $J_1$ , and Laguerre, for  $q^2$  and  $J_2$ , polynomials:

$$P_{2D}(\zeta, q^2, J_1, J_2) = G_{2D} \left[ 1 + \sum_{n=3}^{\infty} \sum_{i,j,k,l=0}^{i+2j+k+2l=n} \frac{(-1)^{j+l}}{i! j! k! l!} \left\langle \zeta^i q^{2j} J_1^k J_2^l \right\rangle_{GC} H_i(\zeta) L_j(q^2) H_k(J_1) L_l(J_2) \right], \quad (4)$$

where  $\sum_{i,j,k,l=0}^{i+2j+k+2l=n}$  stands for summation over all combinations of non-negative  $i, j, k, l$  such that  $i + 2j + k + 2l$  adds to the order of the expansion term  $n$ . The coefficients of the expansion are combinations of moments

$$\left\langle \zeta^i q^{2j} J_1^k J_2^l \right\rangle_{GC} = \frac{j! l!}{(-1)^{j+l}} \left\langle H_i(\zeta) L_j(q^2) H_k(J_1) L_l(J_2) \right\rangle$$

that vanish in the Gaussian limit. In the lowest  $n = 3$  order they coincide with the cumulants of our variables.

In ND the Gaussian JPDF,  $G_{ND}(\zeta, q^2, J_1, J_{s \geq 2})$ , retains complete factorization with respect to  $\zeta, q^2, J_1$

$$G_{ND} = \frac{\left(\frac{N}{2}\right)^{\frac{N}{2}} q^{N-2}}{2\pi \Gamma\left[\frac{N}{2}\right]} \exp \left[ -\frac{\zeta^2}{2} - \frac{N q^2}{2} - \frac{J_1^2}{2} \right] \mathcal{G}(J_{s \geq 2}) \quad (5)$$

so in the moment expansion this sector always gives rise to Hermite  $H_i(\zeta)$ ,  $H_k(J_1)$  and generalized Laguerre  $L_j^{(N-2)/2}(N q^2/2)$  polynomials. However, the distributions of the rest  $J_{s \geq 2}$  contained in  $\mathcal{G}(J_{s \geq 2})$  are coupled.

Specifically, in 3D,  $\mathcal{G}(J_2, J_3) = \frac{25\sqrt{5}}{6\sqrt{2\pi}} \exp[-\frac{5}{2} J_2]$  and  $J_3$

is distributed uniformly between  $-J_2^{3/2}$  and  $J_2^{3/2}$ . Let us denote the orthogonal polynomials in these two variables,  $F_{lm}(J_2, J_3)$ , where  $l$  is the power of  $J_2$  and  $m$  is the power of  $J_3$ . They obey the orthonormality condition

$$\int_0^\infty dJ_2 \int_{-J_2^{3/2}}^{J_2^{3/2}} dJ_3 \mathcal{G} F_{lm}(J_2, J_3) F_{l'm'}(J_2, J_3) = \delta_{ll'} \delta_{mm'}$$

We shall not give the full theory of these polynomials here, but note that one can construct them by a Gram-Schmidt orthogonalization procedure to any given order. Two special cases  $F_{l0} = \sqrt{\frac{3 \times 2^l \times l!}{(3+2l)!}} L_l^{(3/2)}\left(\frac{5}{2} J_2\right)$  and  $F_{01} = \frac{5}{\sqrt{21}} J_3$  are sufficient to obtain the general expression for the Euler characteristic of the excursion sets of the field to arbitrary order, and to calculate the critical point and skeleton statistics to quartic order.

Hence, we write the moment expansion for invariant non-Gaussian JPDF,  $P_{3D}(\zeta, q^2, J_1, J_2, J_3)$  as a series in the power order of the field,  $n$  in the form

$$\begin{aligned} P_{3D} = G_{3D} & \left[ 1 + \sum_{n=3}^{\infty} \sum_{i,j,k,l=0}^{i+2j+k+2l=n} \frac{(-1)^{j+l} 3^j 5^l \times 3}{i! (1+2j)! k! (3+2l)!} \left\langle \zeta^i q^{2j} J_1^k J_2^l \right\rangle_{GC} H_i(\zeta) L_j^{(1/2)}\left(\frac{3}{2} q^2\right) H_k(J_1) L_l^{(3/2)}\left(\frac{5}{2} J_2\right) \right. \\ & + \sum_{n=3}^{\infty} \sum_{i,j,k=0}^{i+2j+k+3=n} \frac{(-1)^j 3^j \times 25}{i! (1+2j)! k! \times 21} \left\langle \zeta^i q^{2j} J_1^k J_3 \right\rangle_{GC} H_i(\zeta) L_j^{(1/2)}\left(\frac{3}{2} q^2\right) H_k(J_1) J_3 \\ & \left. + \sum_{n=5}^{\infty} \sum_{i,j,k,l=0, m=1}^{i+2j+k+2l+3m=n} \frac{(-1)^j 3^j c_{lm}}{i! (1+2j)! k!} \left\langle \zeta^i q^{2j} J_1^k J_2^l J_3^m \right\rangle_{GC} H_i(\zeta) L_j^{(1/2)}\left(\frac{3}{2} q^2\right) H_k(J_1) F_{lm}(J_2, J_3) \right]. \quad (6) \end{aligned}$$

The first high order term contains  $J_2$  but not  $J_3$ , the second is linear in  $J_3$  and contains no  $J_2$ , while for where we left the normalization coefficient  $c_{lm}$  undetermined, contains all the remaining combinations of  $J_2$  and  $J_3$ . In this last term,  $c_{01} = 0$ , so that the first contribution to it,  $l = m = 1$ , is of the fifth power of the field  $\propto J_2 J_3$ . The moment combinations that give the expansion coefficients are found analogously to 2D case.

The formulas (4) and (6) provide the joint probability function of a field and its derivatives up to second order in terms of the invariant variables to an arbitrary order in the moment expansion. They allow to easily compute any rotation-invariant statistics that depend exclusively on these descriptors of the field. In particular, one can compute the Euler characteristic of the excursion sets of the field analytically, the density of the extrema, and the properties of the critical lines that describe the skeleton of the field, as we describe in detail in the follow-up paper.

Determining the topological Euler characteristic,  $\chi$ , and the density of the extrema in the high excursion sets  $x > \nu$  of the random field are two related classical problems in the study of the geometry of a random field. Both are reduced [1] to the evaluation under the

extremal condition  $q^2 = 0$  of the statistical average of the Gaussian curvature of the field surface given by the invariant  $I_N$ , the difference being the various conditions set on the signs of the eigenvalues of the Hessian. For the Euler characteristic integration is unconstrained

$$\frac{\chi(\nu)}{2} = (-1)^N \int_{\nu}^{\infty} dx \int dq^2 q^{N-1} \delta_D^N(q^2) \int \prod_{s=1}^N dJ_s P_{ND} I_N,$$

while to find the number density of extremal points of different types one integrates over the regions in  $J_s$  space where particular signature of the eigenvalues is maintained. Since  $I_s$  are just low order polynomials of the  $J_s$  variables, Eqs (4) and (6) are well suited to perform the required calculations.

The Euler characteristic can be computed completely by noting from Eq. (2) that  $I_N$  depends only linearly on  $J_{s \geq 2}$  (e.g.,  $I_3 = \frac{1}{27}(J_1^3 - 3J_1J_2 + 2J_3)$  in 3D), hence all terms in JPDF of higher order in  $J_{s \geq 2}$  do not contribute. The 2D and 3D results [18] can be combined in a very compact form if one re-expresses the coefficients back in terms of the field  $x$  itself and the invariants  $I_s$

$$\begin{aligned} \chi(\nu) = & \frac{1}{2} \text{Erfc} \left( \frac{\nu}{\sqrt{2}} \right) \chi(-\infty) + \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\nu^2}{2} \right) \times \frac{2}{(2\pi)^{N/2}} \left( \frac{\gamma}{\sqrt{N}} \right)^N \left[ H_{N-1}(\nu) + \right. \\ & \left. + \sum_{n=3}^{\infty} \sum_{s=0}^N \gamma^{-s} \sum_{i,j=0}^{i+2j=n-s} \frac{(-N)^{j+s} (N-2)!! L_j^{(\frac{N-2}{2})}(0)}{i!(2j+N-2)!!} \left\langle x^i q^{2j} I_s \right\rangle_{GC} H_{i+N-s-1}(\nu) \right], \end{aligned} \quad (7)$$

where  $i = 0, s = N$  terms have been combined into the boundary term  $\propto \chi(-\infty)$  fixed by the topology of the manifold and should be omitted from the sum.  $I_0 \equiv 1$ .

Calculation of the extrema number density produces an analytical result only in 2D. Even then the result to fourth order is already too complicated to be reproduced here and is deferred to [13]. Instead here we demonstrate our results on two model non-Gaussian fields.

For the a 2D case we generate a Gaussian random field  $x_G$  with the scale-invariant power spectrum  $\propto k^{-1.5}$ , smoothed with a Gaussian filter of 5-pixel width. The underlying grid is then displaced by  $\alpha \nabla \Delta^{-1} x_G$  and the initial field is resampled on this grid, producing a one-parameter non-Gaussian field,  $x^{(\alpha)}$ . This toy model is able to produce high level of non-Gaussianity without generating high field excursions, as demonstrated in Fig. 1 by a  $2048^2$  realization of the field with  $\alpha = 0.5$ .

In 3D, we use scale-invariant ( $\propto k^{-1}$ ) simulations of cosmological density evolved to the mildly non-linear stage of gravitational instability where the variance of inhomogeneities is  $\sigma = 0.1$  of the mean density.

In Fig. 2 we plot the extrema count [19] and the Euler

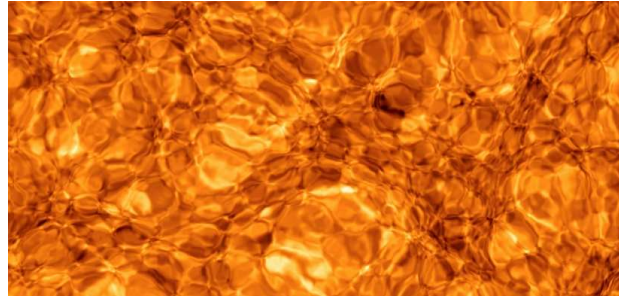


FIG. 1: A section of strongly non-Gaussian 2D field.

characteristic computed within our formalism to 3rd and 4th order respectively, using the moments measured numerically in our toy simulations. The *left* column demonstrates the avenues our formalism opens for the theoretical study of critical points of non-Gaussian fields. The *right* column shows, on the example of the Euler characteristic, the importance of accounting for high moments of the expansion for some models of non-Gaussianity, as well as the convergence process of the expansion for the

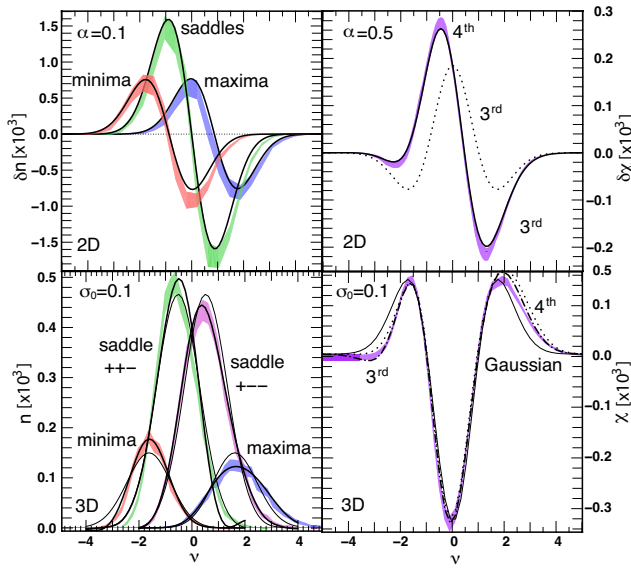


FIG. 2: The number of extrema (left) and the Euler characteristic (right) for the 2D toy model (top) and 3D gravitational collapse model (bottom). In dimensional units both quantities are given per  $R_*^N$  volume defined as in [11]. The shaded bands correspond to  $2\sigma$  variations over the mean measurements of 100 (2D) or 25 (3D) realizations, while the curves give the 3<sup>rd</sup> and 4<sup>th</sup> order predictions. A higher order correction is required to fit the strong non-Gaussian field shown in Fig. 1. In 2D only the correction to the Gaussian limit is shown. In 3D, the Gaussian prediction is shown as a thin line.

gravitational instability models.

The ability to analyze the full series of the moment expansion for geometrical statistics is important from sev-

eral points of view. In cosmological applications, e.g. a mildly non-linear gravitational instability as simulated in 3D in Figure 2, or lensing of CMB maps, one deals with perturbative deviation from Gaussianity governed by the small value of the variance  $\sigma$ . In this case Gram-Charlier series can be rearranged in (asymptotic) Edgeworth [14] power series in  $\sigma$  [20] and the full expansion can be used to obtain an increasingly accurate description of the statistics, and conduct estimates of residual errors. Outside of established cosmological perturbation theory, theoretical formalism that deals globally with all the terms is all the more important. It may allow us to design a customized truncation criteria, which may not be uniform across all the statistics, or even for all the partial contributions to a given statistical descriptor [21].

In the cosmological context, non-Gaussianity of extragalactic fields can be used to constrain the dark energy equation of state via 3D galactic surveys, or shed light on the physics of the early Universe through 2D CMB maps. The predictions of theoretical models are often given through the hierarchy of the differences between the moments to their Gaussian limit. Higher order moments are generally difficult to test directly in real-life observations, due to their sensitivity to very rare events. The geometrical analysis of the critical events in the field provides more robust measures of non-Gaussianity and is becoming an active field of investigation [15, 16]. The present paper advances the formal development of this subject, opening ways to expand the list and sharpen statistical tools that may detect unique signatures of fundamental processes in our Universe.

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  - [17] By Gaussian JPFD we mean the limit of the Gaussian field. Note that even in this limit, the rotation invariant variables themselves are not, in general, Gaussian.
  - [18] We conjecture that Eq. (7) is valid for all  $N$ .
  - [19] Extrema counts are obtained by integrating analytically Eq. (4) in 2D and numerically Eq. (6) in 3D over the relevant bounds of eigenvalues [11]
  - [20] It is worth noting that leading  $n = 3$  polynomial term coincides with linear in  $\sigma$  correction to PDF.
  - [21] e.g. Eq. (7) can be rearranged as series in the Hermite polynomials order, where coefficients are (asymptotic) series in the moments of the field. Convergence properties and the best truncation may differ for each coefficient.